Igusa's Modular Form and the Classification of Siegel Modular Threefolds

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0 Introduction

For an integer $d \geq 1$ let

$$E_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad \Lambda_d = \begin{pmatrix} 0 & E_d \\ -E_d & 0 \end{pmatrix}.$$

We consider the symplectic group

$$\tilde{\Gamma}_{1,d} := \mathrm{Sp}(\Lambda_d, \mathbb{Z}).$$

For $d = 1$ this is the usual integer symplectic group $\mathrm{Sp}(4, \mathbb{Z})$. The group $\tilde{\Gamma}_{1,d}$ operates on the *Siegel space* of genus 2

$$\mathbb{H}_2 = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}); \mathrm{Im}\ \tau > 0 \right\}$$

by

$$\tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} : \tau \mapsto (\tilde{A}\tau + \tilde{B}E_d)(\tilde{C}\tau + \tilde{D}E_d)^{-1}E_d.$$

The quotient

$$\mathcal{A}_{1,d} = \tilde{\Gamma}_{1,d} \backslash \mathbb{H}_2$$

is the moduli space of $(1,d)$-polarized abelian surfaces. Alternatively we can consider the following subgroup of the usual rational symplectic group $\mathrm{Sp}(4, \mathbb{Q})$. Let

$$R_d = \mathrm{diag}(1, 1, 1, d)$$

and set

$$\Gamma_{1,d} := R_d^{-1} \tilde{\Gamma}_{1,d} R_d \subset \mathrm{Sp}(4, \mathbb{Q}).$$

Then $\Gamma_{1,d}$ acts in the usual way on $\mathbb{H}_2$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}$$

and

$$\mathcal{A}_{1,d} = \tilde{\Gamma}_{1,d} \backslash \mathbb{H}_2 = \Gamma_{1,d} \backslash \mathbb{H}_2.$$

Let $L = \mathbb{Z}^4$ be the lattice on which Λ_d defines a symplectic form and let $L^\vee$ be the dual lattice of L. We consider the following subgroups of $\tilde{\Gamma}_{1,d}$ defined by

$$\begin{aligned} \tilde{\Gamma}_{1,d}^{\mathrm{lev}} &:= \left\{ M \in \tilde{\Gamma}_{1,d}; \quad M|_{L^\vee/L} = \mathrm{id} \right\} \\ \tilde{\Gamma}_{1,d}(n) &:= \left\{ M \in \tilde{\Gamma}_{1,d}; \quad M \equiv \mathbf{1} \bmod n \right\} \quad (n \geq 1) \\ \tilde{\Gamma}_{1,d}^{\mathrm{lev}}(n) &:= \tilde{\Gamma}_{1,d}^{\mathrm{lev}} \cap \tilde{\Gamma}_{1,d}(n). \end{aligned}$$

This gives rise to subgroups of $\mathrm{Sp}(4,\mathbb{Q})$:

$$\begin{array}{rcl}\Gamma^{\mathrm{lev}}_{1,d} & := & R_d^{-1}\tilde{\Gamma}^{\mathrm{lev}}_{1,d}R_d \\ \Gamma_{1,d}(n) & := & R_d^{-1}\tilde{\Gamma}_{1,d}(n)R_d \\ \Gamma^{\mathrm{lev}}_{1,d}(n) & := & R_d^{-1}\tilde{\Gamma}^{\mathrm{lev}}_{1,d}(n)R_d = \Gamma^{\mathrm{lev}}_{1,d}\cap\Gamma_{1,d}(n),\end{array}$$

resp. to the moduli spaces

$$\begin{array}{rcccl}\mathcal{A}^{\mathrm{lev}}_{1,d} & = & \tilde{\Gamma}^{\mathrm{lev}}_{1,d}\backslash\mathbb{H}_2 & = & \Gamma^{\mathrm{lev}}_{1,d}\backslash\mathbb{H}_2 \\ \mathcal{A}_{1,d}(n) & = & \tilde{\Gamma}_{1,d}(n)\backslash\mathbb{H}_2 & = & \Gamma_{1,d}(n)\backslash\mathbb{H}_2 \\ \mathcal{A}^{\mathrm{lev}}_{1,d}(n) & = & \tilde{\Gamma}^{\mathrm{lev}}_{1,d}(n)\backslash\mathbb{H}_2 & = & \Gamma^{\mathrm{lev}}_{1,d}(n)\backslash\mathbb{H}_2.\end{array}$$

The geometric meaning of these moduli spaces is the following:

$$\begin{aligned}\mathcal{A}^{\mathrm{lev}}_{1,d} = \{(A,H,\alpha); (A,H) &\text{ is a } (1,d)\text{-polarized abelian surface,}\\ &\alpha \text{ is a canonical level-structure}\}.\end{aligned}$$

Here a *canonical* level-structure is a symplectic basis of the kernel of the map $\lambda_H : A \to \hat{A} = \mathrm{Pic}^0 A$. (Note that this kernel is (non-canonically) isomorphic to $\mathbb{Z}/d \times \mathbb{Z}/d$.) Similarly

$$\begin{aligned}\mathcal{A}_{1,d}(n) = \{(A,H,\beta); (A,H) &\text{ is a } (1,d)\text{-polarized abelian surface,}\\ &\beta \text{ is a full level-n structure}\}.\end{aligned}$$

Here a *full level-*n *structure* is a symplectic basis of the group $A^{(n)}$ of n-torsion points of A. Finally

$$\begin{aligned}\mathcal{A}^{\mathrm{lev}}_{1,d}(n) = \{(A,H,\alpha,\beta); (A,H) &\text{ is a } (1,d)\text{-polarized abelian surface,}\\ \alpha \text{ is a canonical level structure, } &\beta \text{ is a full level-}n\text{ structure}\}.\end{aligned}$$

Note that

$$\tilde{\Gamma}^{\mathrm{lev}}_{1,d}/\tilde{\Gamma}_{1,d} \cong \Gamma^{\mathrm{lev}}_{1,d}/\Gamma_{1,d} \cong \mathrm{SL}(2,\mathbb{Z}/d)$$

and that we have, therefore, a Galois covering $\mathcal{A}^{\mathrm{lev}}_{1,d} \to \mathcal{A}_{1,d}$ with Galois group $\mathrm{PSL}(2,\mathbb{Z}/d)$.

The aim of this short note is to prove two results about the classification of these Siegel modular varieties.

Theorem 0.1 *$\mathcal{A}_{1,d}(n)$ is of general type if $(d,n)=1$ and $n \geq 4$.*

Since $\mathcal{A}_{1,1}(n)$ is rational for $n \leq 3$ this is the best result which one can hope for if one considers all d simultaneously. The space $\mathcal{A}_{1,3}(2)$ has a Calabi-Yau model ([BN], [GH]) and hence Kodaira dimension 0, whereas $\mathcal{A}_{1,3}(3)$ is of general type ([GH, Theorem 3.1]). For prime numbers p Sankaran [S] has proved that $\mathcal{A}_{1,p}$ is of general type for $p \geq 173$. A similar result for $\mathcal{A}_{1,d}$, where d is not necessarily prime, is, as for as I know, not known. Borisov [Bo] has shown that, up to conjugation, there are only finitely many subgroups Γ of $\mathrm{Sp}(4,\mathbb{Z})$ such that $\mathcal{A}(\Gamma) = \Gamma\backslash\mathbb{H}_2$ is not of

general type. Recall however, that the groups $\Gamma_{1,d}(n)$ are not subgroups of $\mathrm{Sp}(4,\mathbb{Z})$ unless d divides n and that, in general, they are also not conjugate to subgroups of $\mathrm{Sp}(4,\mathbb{Z})$. An essential ingredient in the proof of the theorem is Igusa's modular form Δ_{10}.

The above theorem is a result about the birational classification of these varieties. If one wants to ask more precise questions, such as whether K is ample, then one has to specify the compactification with which one wants to work.

Theorem 0.2 *The Voronoi compactification $\left(\mathcal{A}^{\mathrm{lev}}_{1,p}(n)\right)^*$ for a prime number p with $(p,n)=1$ is smooth and has ample canonical bundle (i.e. is a canonical model in the sense of Mori theory) if and only if $n \geq 5$.*

Here a few words are in order: By Voronoi compactification we mean the compactification defined by the second Voronoi decomposition. We choose this compactification because Alexeev [A1] has shown that it appears naturally when one wants to construct a toroidal compactification which represents a geometrically meaningful functor. The addition of a canonical level structure has two reasons: The spaces $(\mathcal{A}_{1,p}(n))^*$ have non-canonical singularities for infinitely many p and n. These singularities come from the toroidal construction, not from fixed points of the group which is neat for $n \geq 3$. Moreover, it is necessary to introduce at least some kind of level structure to obtain a functorial description of the compactifications. A canonical level structure will be sufficient for this [A2]. Finally the restriction to prime numbers p in Theorem 0.2 is done to keep the technical difficulties to an acceptable level. I believe that this restriction is not essential. This result also supports a conjecture made in [H] for principal polarizations. This is in so far interesting as the case treated here cannot, as it could be in the case $p=1$, be easily derived from known results on $\mathcal{M}_2$.

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1 General type

In this section we want to prove

Theorem 1.1 *If $(d,n)=1$ and $n \geq 4$ then $\mathcal{A}_{1,d}(n)$ is of general type.*

We shall work with the Voronoi compactification $\mathcal{A}^*_{1,d}(n)$ of $\mathcal{A}_{1,d}(n)$. Before we can prove the theorem we need to know something about the coordinates of $\mathcal{A}^*_{1,d}(n)$ near a cusp. Recall that the codimension 1 cusps are given by lines $l \subset \mathbb{Q}^4$ up to the action of the group $\Gamma_{1,d}(n)$ and that the codimension 2 cusps are similarly given by isotropic planes $h \subset \mathbb{Q}^4$. For any

such l, resp. h and any group Γ we denote the lattice part of the stabilizer of l, resp. h in Γ by $P'_\Gamma(l)$, resp. $P'_\Gamma(h)$. These lattices have rank 1, resp. 3.

Lemma 1.2 (i) *For every line $l \subset \mathbb{Q}^4$ there is an inclusion $P'_{\mathrm{Sp}(4,\mathbb{Z})}(l) \subset P'_{\Gamma_{1,d}}(l)$ with cokernel $\mathbb{Z}/d_1$ where $d_1|d$.*

(ii) *For every isotropic plane $h \subset \mathbb{Q}^4$ there is an inclusion $P'_{\mathrm{Sp}(4,\mathbb{Z})}(h) \subset P'_{\Gamma_{1,d}}(h)$ with cokernel $\mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \mathbb{Z}/d_3$ where $d_i|d$ for $i = 1,2,3$.*

Proof. (i) Let l be a line which corresponds to a given cusp of $\Gamma_{1,d}$. By [FS, Satz 2.1] we may assume that

$$l = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} l_0 \quad , \quad \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} =: M \in \mathrm{Sp}(4,\mathbb{Z})$$

where $l_0 = e_3 = (0,0,1,0)$. The group $Q'(l_0) = M^{-1}P'_{\Gamma_{1,d}}(l)M$ is a rank 1 lattice which fixes l_0. We want to compare this to the rank 1 lattice

$$P'_{\mathrm{Sp}(4,\mathbb{Z})}(l_0) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; s \in \mathbb{Z} \right\} \subset \mathrm{Sp}(4,\mathbb{Z}).$$

Recall from [HKW, Proposition I.1.16] that every element g in $\Gamma_{1,d}$ fulfills the following congruences

$$g - \mathbf{1} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & d\mathbb{Z} \\ d\mathbb{Z} & \mathbb{Z} & d\mathbb{Z} & d\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & d\mathbb{Z} \\ \mathbb{Z} & \frac{1}{d}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

Hence

$$\begin{pmatrix} {}^tD & 0 \\ -{}^tC & {}^tA \end{pmatrix} g \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} * & S \\ * & * \end{pmatrix}, \quad S \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

In particular $P'_{\mathrm{Sp}(4,\mathbb{Z})}(l_0)$ is contained in $Q'(l_0)$. Hence $P'_{\Gamma_{1,d}}(l)/P'_{\mathrm{Sp}(4,\mathbb{Z})}(l) \cong \mathbb{Z}/d_1$ for some d_1. The claim $d_1|d$ follows since

$$M \begin{pmatrix} 1 & 0 & d & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M^{-1} \in P'_{\Gamma_{1,d}}(l).$$

(ii) Again we can choose an element $M \in \mathrm{Sp}(4,\mathbb{Z})$ such that $h = M(h_0)$ where $h_0 = e_3 \wedge e_4$. Then

$$Q'(h_0) = M^{-1}P'_{\Gamma_{1,d}}(h)M$$

consists of elements of the form

$$\begin{pmatrix} 1 & 0 & d_1\mathbb{Z} & d_2\mathbb{Z} \\ 0 & 1 & d_2\mathbb{Z} & d_3\mathbb{Z} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By (i) we can conclude that $d_1, d_3 \in \mathbb{N}$. We claim that also $d_2 \in \mathbb{N}$. To prove this recall that there is a sublattice $L_0 \subset L = \mathbb{Z}^4$ with $L/L_0 \cong \mathbb{Z}/d$ such that $\Gamma_{1,d}(L_0) \subset L$. (This is simply the lattice spanned by e_1, e_2, e_3, de_4). Hence the same statement must be true for $M^{-1}P'_{\Gamma_{1,d}}(h)M$, but this implies that $d_2 \in \mathbb{N}$. The assertions $d_1|d$ and $d_3|d$ follow from (i) and $d_2|d$ follows again since

$$M \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & d & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M^{-1} \in P'_{\Gamma_{1,d}}(h).$$

□

Proof of the theorem. We consider the following maps of moduli spaces

$$\begin{tikzcd} & \mathcal{A}_{1,d}^{\mathrm{lev}} \arrow[dl] \arrow[dr] & & \mathcal{A}_{1,d}(n) \arrow[dl] \\ \mathcal{A}_{1,1} & & \mathcal{A}_{1,d} & \end{tikzcd}$$

The map $\mathcal{A}_{1,d}^{\mathrm{lev}} \to \mathcal{A}_{1,1}$ comes from the inclusion $\Gamma_{1,d}^{\mathrm{lev}} \subset \mathrm{Sp}(4,\mathbb{Z})$. (The argument given in [HKW, Proposition I.1.20] for d prime goes through unchanged for all d.). Note that $\Gamma_{1,d}^{\mathrm{lev}}$ is not normal in $\mathrm{Sp}(4,\mathbb{Z})$ and hence $\mathcal{A}_{1,d}^{\mathrm{lev}} \to \mathcal{A}_{1,1}$ is not Galois. The other maps $\mathcal{A}_{1,d}^{\mathrm{lev}} \to \mathcal{A}_{1,d}$ and $\mathcal{A}_{1,d}(n) \to \mathcal{A}_{1,d}$ are Galois covers.
An essential ingredient in the proof is Igusa's modular form

$$\Delta_{10} = \prod_{m \text{ even}} \Theta_m^2(\tau)$$

given by the product of the squares of all even theta null values. This is a cusp form of weight 10 with respect to $\mathrm{Sp}(4,\mathbb{Z})$. In fact it is, up to scalar, the unique weight 10 cusp form with respect to $\mathrm{Sp}(4,\mathbb{Z})$. Recall that it vanishes exactly along the $\mathrm{Sp}(4,\mathbb{Z})$-translates of the diagonal

$$\mathbb{H}_1 \times \mathbb{H}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}; \mathrm{Im}\ \tau_1,\ \mathrm{Im}\ \tau_3 > 0 \right\} \subset \mathbb{H}^2$$

where it vanishes of order 2. Since $\Gamma_{1,d}^{\mathrm{lev}}$ is a subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ we can also consider Δ_{10} as a cusp form with respect to $\Gamma_{1,d}^{\mathrm{lev}}$. Recall that for any

modular form G and a matrix M the slash-operator is defined by

$$G|_k M := \det(C\tau + D)^{-k}\; G(M\tau) \quad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right).$$

We consider the multiplicative symmetrization

$$F_0 := \prod_{M \in \mathrm{PSL}(2,\mathbb{Z}/d)} \Delta_{10}|_{10} M.$$

It is straightforward to check that F_0 is a cusp form with respect to $\Gamma_{1,d}$ of weight $10\mu(d)$ where

$$\mu(d) = \frac{1}{2}|\mathrm{SL}(2,\mathbb{Z}/d)| = \frac{1}{2} d^3 \prod_{p|d}(1 - \frac{1}{p^2}) \quad (d \geq 3),$$

resp. $\mu(2) = 6$. Clearly we can also consider F_0 as a cusp form with respect to the smaller group $\Gamma_{1,d}(n)$. Let L be the ($\mathbb{Q}$-)line bundle of modular forms of weight 1. By abuse of notation we shall use the same notation for whatever moduli space we are considering.

Claim 1: For every point P on the boundary of $\mathcal{A}^*_{1,d}(n)$ the modular form F_0 defines an element in $m_P^{n\mu(d)} L^{10\mu(d)}$.

For points on the codimension 1 cusps this follows immediately from Lemma 1.2 (i) and $(n,d) = 1$. To prove it in general we consider an isotropic plane h and the lattices $N := P'_{\Gamma_{1,d}}(h)$ and $N' := P'_{\Gamma_{1,d}(n)}(h)$. Let $\sigma \in \Sigma_{\mathrm{vor}}$ be a 3-dimensional cone and let $T_\sigma(N)$, resp. $T_\sigma(N')$ be the corresponding affine parts in the toric variety $T_{\Sigma_{\mathrm{vor}}}(N)$, resp. $T_{\Sigma_{\mathrm{vor}}}(N')$. We claim that Δ_{10} defines a function on the closure of the image of $P'_{\Gamma_{1,d}}(h)\backslash\mathbb{H}_2$ in $T_\sigma(N)$. First of all Δ_{10} is a function on $P'_{\Gamma_{1,d}}\backslash\mathbb{H}_2$ by Lemma 1.2 (ii). Since $T_\sigma(N)$ is normal, it is enough to show that this function extends to the codimension 1 boundary components. This follows from Lemma 1.2 (i). Since Δ_{10} is a cusp form it follows that $\Delta_{10} \in m_P$ for every point P on the boundary. By construction of F_0 this gives claim 1 in the case $n = 1$. Since $(d,n) = 1$ we have $N' = nN$ and this gives the claim for general n.

Let $\mathcal{A}^*_{1,d}(n)$ be the Voronoi compactification of $\mathcal{A}_{1,d}(n)$, i.e. the toroidal compactification given by the second Voronoi decomposition of the cone of semi-positive definite symmetric real (2×2) -matrices (in [HKW] this was also called Legendre decomposition). If $n \geq 3$ the group $\Gamma_{1,d}(n)$ is neat (this follows from a general result of Serre which says that every algebraic integer which is a unit and which is congruent to 1 mod $n (n \geq 3)$ is equal to 1). In particular $\mathcal{A}_{1,d}(n)$ is smooth. The toroidal compactification $\mathcal{A}^*_{1,d}(n)$ will, in general, however have singularities. These arise because the fan given by the Voronoi decomposition is not always basic, i.e. there may be cones which are not spanned by elements of a basis of the lattice. We can always choose

a suitable subdivision of the fan given by the Voronoi decomposition and in this way construct a smooth resolution $\psi : \tilde{\mathcal{A}}_{1,d}(n) \to \mathcal{A}^*_{1,d}(n)$ such that the exceptional divisor is a normal crossing divisor.
Let $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$. It is well known that, if G is a weight $3k$ cusp form which vanishes of order $\geq k$ along all 1-codimensional cusps, then $G\omega^k$ defines a k-fold canonical form on the smooth part of $\mathcal{A}^*_{1,d}(n)$.

Claim 2: The space of k-fold canonical forms which extends to the smooth part of $\mathcal{A}^*_{1,d}(n)$ grows (at least for sufficiently divisible k) as ck^3 for some postive constant c.

To prove this claim recall that $K = 3L - D$ on the smooth part of $\mathcal{A}^*_{1,d}(n)$, where L is the ($\mathbb{Q}$-) line bundle of modular forms of weight 1 and D is the boundary, i.e. the union of all 1-codimensional cusps. By claim 1 the form F_0 gives the equality

$$10\mu(d)L = n\mu(d)D + D_{\text{eff}}$$

for some effective divisor D_{eff} on the smooth part of $\mathcal{A}^*_{1,d}(n)$. ¿From this we obtain

$$-D = -\frac{10}{n}L + \frac{1}{n\mu(d)}D_{\text{eff}}.$$

Combining this equality with the expression for K gives us

$$K = \left(3 - \frac{10}{n}\right)L + \frac{1}{n\mu(d)}D_{\text{eff}}.$$

For $n \geq 4$ the factor in front of L is positive and the claim follows since $h^0(L^k)$ grows as ck^3.

Claim 3: If $F_{3k}\omega^k$ defines a k-fold canonical form on the smooth part of $\mathcal{A}^*_{1,d}(n)$ then $\left(F_0^3\omega^{10\mu(d)}\right)^k \left(F_{3k}\omega^k\right)$ extends to $\tilde{\mathcal{A}}_{1,d}(n)$. We first notice that, since $n \geq 4$, the form $F_0^3\omega^{10\mu(d)}$ extends to the smooth part of $\mathcal{A}^*_{1,d}(n)$. The following part of the argument follows closely the proof of [S, Theorem 6.3]. It is enough to prove that the form in question extends to the generic point of each component of the exceptional divisor. Let E be a component of the exceptional divisor of the resolution $\tilde{\mathcal{A}}_{1,d}(n) \to \mathcal{A}^*_{1,d}(n)$. It is enough to consider points which lie on only one boundary component. We can choose local analytic coordinates z_1, z_2, z_3 on an open set U such that $E = \{z_1 = 0\}$. Recall that U is an open set in some toroidal variety $T_{\tilde{\Sigma}}(N')$ where $N' = P'_{\Gamma_{1,d}(n)}(h)$ for some isotropic plane h and $\tilde{\Sigma}$ is a refinement of the fan Σ_{Vor} defined by the Voronoi decomposition. Moreover the coordinates of the torus are of the form $t_i = e^{2\pi i a_i \tau_i}$ for some rational numbers a_i. A local equation for E is given by $t_1^{b_1}t_2^{b_2}t_3^{b_3}$ for suitable b_i and hence we can set $z_1 = t_1^{b_1}t_2^{b_2}t_3^{b_3}$. Since $\partial z_1/\partial\tau_j = 2\pi i a_j b_j z_1$ we can conclude that the order of

$J = \det(\partial\tau_i/\partial z_j)$ along E is $v_E(J) \geq -1$. It follows again from claim 1 that

$$v_E\left(F_0^3 J^{10\mu(d)}\right) \geq (3n-10)\mu(d).$$

Therefore $\left(F_0^3\omega^{10\mu(d)}\right)$ defines a section of $\mu(d)(10K - (3n-10)E)$. By assumption $F_{3k}\omega^k$ defines a section of $\psi^*\left(kK_{\mathcal{A}^*_{1,d}(n)}\right) = k(K - \alpha E)$ where α is the discrepancy of E. Altogether $\left(F_0^3\omega^{10\mu(d)}\right)^k\left(F_{3k}\omega^k\right)$ defines a section of

$$\begin{aligned}(10\mu(d)kK - k\mu(d)(3n-10)E) + k(K-\alpha E) = \\ = k\left[(10\mu(d)+1)K - (\mu(d)(3n-10)+\alpha)E\right].\end{aligned}$$

All singularities here are cyclic quotient singularities. This follows from Lemma 1.2 and the fact that $T_{\Sigma_{\mathrm{vor}}}(P'_{\mathrm{Sp}(4,\mathbb{Z})}(h))$ is smooth. Hence the singularities are log-terminal, i.e. $\alpha > -1$. This implies that $\mu(d)(3n-10)+\alpha > 0$ for $n \geq 4$ and thus the claim follows.

The theorem now follows easily from by combining claim 2 and claim 3. $\square$

2 Ample canonical bundle

It is the aim of this section to prove the following

Theorem 2.1 *Let p be an odd prime number and assume that $(n,p) = 1$. The Voronoi compactification $(\mathcal{A}^{\mathrm{lev}}_{1,p}(n))^*$ is smooth and has ample canonical bundle if and only if $n \geq 5$.*

We remark that this result is also known to be true for $p = 1$ (cf.[Bo], [H]). Before we prove this theorem we recall the geometry of the spaces $(\mathcal{A}^{\mathrm{lev}}_{1,p})^*$ which was described in detail in [HKW]. The Tits building of the group $\Gamma^{\mathrm{lev}}_{1,p}$ consists of $1+(p^2-1)/2$ lines and $p+1$ isotropic planes. The lines consist of one so-called central line and $(p^2-1)/2$ peripheral lines. If $D(l_0)$ is the closed boundary surface which belongs to the central line, then there is a map $K(p) \to D(l_0)$ which is an immersion, but not an embedding if $p > 3$. Here $K(p)$ is the Kummer modular surface of level p, i.e. the quotient of Shioda's modular surface $S(p)$ by the involution which acts by $x \mapsto -x$ on every fibre. For each peripheral boundary component $D(l)$ there exists an isomorphism $K(1) \cong D(l)$ where $K(1)$ is the Kummer modular surface of level 1.

If we add a level-n structure clearly the number of inequivalent cusps will increase. We shall, however, still speak about central or peripheral cusps with respect to $\Gamma^{\mathrm{lev}}_{1,p}(n)$ depending on whether this defines a central or peripheral cusp with respect to $\Gamma^{\mathrm{lev}}_{1,p}$. Now assume $n \geq 3$. Then one shows exactly as in the proof of [HKW, Theorem I.3.151] that there are immersions $S(np) \to D(l_c)$ if l_c is a central cusp, resp. $S(n) \to D(l_p)$ if l_p

is a peripheral cusp. The reason why we have Shioda modular surfaces here instead of Kummer modular surfaces is that for $n \geq 3$ the matrix **-1** is not contained in $\Gamma_{1,p}^{\text{lev}}(n)$. It will be immaterial for us whether these maps are immersions or embeddings.

We shall write the boundary as

$$D = \sum_{i \in I} D_c^i + \sum_{j \in J} D_p^j$$

where D_c^i are the central and D_p^j the peripheral boundary components.

We recall the following well known facts about Shioda modular surfaces. For $k \geq 3$ the surface $S(k) \to X(k)$ is the universal elliptic curve with a level-k structure. The base curve $X(k)$ is the modular curve of level k. It has $t(k) = \frac{1}{2}k^2 \prod_{p|k}(1 - \frac{1}{p^2})$ cusps and the fibre of $S(k)$ over the cusps are singular of type I_k, i.e. a k-gon of (-2)-curves. The genus of $X(k)$ equals $1 + (k-6)t(k)/12$ and the line bundle $L_{X(k)}$ of modular forms of weight 1 has degree $kt(k)/12$. The elliptic fibration $\pi : S(k) \to X(k)$ has k^2 sections L_{ij} which form a group $\mathbb{Z}/k \times \mathbb{Z}/k$. By F we denote a general fibre of $S(k)$. It is well known (cf[BH, pp.78-80]) that

$$\begin{aligned} K_{S(k)} &\equiv \frac{k-4}{4} t(k) F, \\ L_{ij}^2 &= \frac{k}{12} t(k) = -\deg L_{X(k)}. \end{aligned}$$

Let $f_c : S(np) \to D_c$, resp. $f_p : S(n) \to D_p$ be the map from $S(np)$ to a central, resp. from $S(n)$ to a peripheral boundary component. Since these maps are immersions we can consider the normal bundles N_c, resp. N_p of these maps.

Proposition 2.2 (i) $N_c \equiv -\dfrac{2}{np} L_{X(np)} - \dfrac{2}{np} \displaystyle\sum_{i,j \in \mathbb{Z}/np \times \mathbb{Z}/np} L_{ij}$

(ii) $N_p \equiv -\dfrac{2}{n} L_{X(n)} - \dfrac{2}{n} \displaystyle\sum_{i,j \in \mathbb{Z}/n \times \mathbb{Z}/n} L_{ij}$.

Proof. We shall give the proof for the central boundary components and indicate how it has to be adopted to the peripheral boundary components. There is a natural action of the group $\sum L_{ij} = \mathbb{Z}/np \times \mathbb{Z}/np$ on $S(np)$. It is an easy calculation to check that this action is induced by elements of $\Gamma_{1,p}^{\text{lev}}(n)/\Gamma_{1,p}$. It follows that N_c is invariant under the group $\mathbb{Z}/np \times \mathbb{Z}/np$ and hence

$$N_c \equiv aF + b \sum L_{ij}$$

for some $a, b \in \mathbb{Q}$ (cf.[BH]). To determine a, b we have to compute the degree of the normal bundle N_c on a general fibre of $S(np)$ and on a section, e.g.

the zero section L_{00}.
As a representative for a central cusp we can take the line $l_0 = (0,0,1,0)$. Assume $(n,p)=1$. We set

$$\Gamma_1'(np) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}); a,d \equiv 1 \bmod np, c \equiv 0 \bmod n, \right. \\ \left. b \equiv 0 \bmod np^2 \right\}.$$

Note that by conjugation with $E = \mathrm{diag}(1,p)$ the group $\Gamma_1'(np)$ is conjugate to the principal subgroup $\Gamma_1(np)$. Then by [HKW, Proposition I.3.98] the stabilizer subgroup $P(l_0)$ of $\Gamma_{1,p}^{\mathrm{lev}}(n)$ is given by

$$P(l_0) = \left\{ \begin{pmatrix} 1 & k & s & m \\ 0 & a & m & b \\ 0 & 0 & 1 & 0 \\ 0 & c & -k & d \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1'(np), k,s \in n\mathbb{Z}, m \in pn\mathbb{Z}\} \right\}.$$

The action of

$$\begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 + s & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$$

gives rise to the partial quotient

$$\begin{array}{ccc} \mathbb{H}_2 & \to & \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1 \\ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} & \mapsto & (t_1 = e^{2\pi i \tau_1/n}, \tau_2, \tau_3). \end{array}$$

The other elements of $P(l_0)$ act as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 - \tau_2(c\tau_3+d)^{-1}c\tau_2 & * \\ \tau_2(c\tau_3+d)^{-1} & (a\tau_3+b)(c\tau_3+d)^{-1} \end{pmatrix},$$

$$\begin{pmatrix} 1 & k & 0 & m \\ 0 & 1 & m & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1' & * \\ \tau_2 + k\tau_3 + m & \tau_3 \end{pmatrix},$$

$$\tau_1' = \tau_1 + k^2\tau_3 + 2k\tau_2 + km.$$

The group

$$P''(l_0) = \left\{ \begin{pmatrix} 1 & k & m \\ 0 & a & b \\ 0 & c & d \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1'(np), k \in n\mathbb{Z}, m \in pn\mathbb{Z} \right\}$$

defines an action on $\mathbb{C}\times\mathbb{H}_1$ by

$$\begin{pmatrix}1 & k & m\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} : (\tau_2,\tau_3)\mapsto(\tau_2+k\tau_3+m,\tau_3)$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & c & d\end{pmatrix} : (\tau_2,\tau_3)\mapsto(\tau_2(c\tau_3+d)^{-1},(a\tau_3+b)(c\tau_3+d)^{-1}).$$

Then $D^0(l_0)=P''(l_0)\backslash\{0\}\times\mathbb{C}\times\mathbb{H}_1$ is the open boundary surface associated to l_0 and conjugation with $E=\operatorname{diag}(1,p)$ shows that $D^0(l^0)\cong S^0(np)$, the open part of $S(np)$ which does not lie over the cusps.
A local equation of $D^0(l_0)$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{H}_1$ is given by $t_1=0$ and hence t_1/t_1^2 is a local section of the conormal bundle. Under the action of the group $P(l_0)$ this transforms as follows:

$$(1)\qquad t_1/t_1^2\mapsto t_1/t_1^2\ e^{2\pi i[-k^2\tau_3-2k\tau_2]/n},$$

$$(2)\qquad t_1/t_1^2\mapsto t_1/t_1^2\ e^{2\pi i\left(-\frac{c\tau_2^2}{c\tau_3+d}\right)/n}.$$

We can use the formulae (1) and (2) to determine the coefficients a and b. We first determine the degree of N_c on a general fibre F. Since $k\in n\mathbb{Z}$, $m\in pn\mathbb{Z}$ the fibre of $S(np)$ over the point $[\tau_3]\in X(np)$ is given by $E_{[\tau_3]}=\mathbb{C}/(\mathbb{Z}n\tau_3+\mathbb{Z}np)$. The standard theta function $\vartheta(\tau_3,\tau_2)$ defines a line bundle of degree n^2p on $E_{[\tau_3]}$ and transforms as follows

$$(3)\ \ \vartheta(\tau_3,\tau_2+k\tau_3+m)=e^{2\pi i[-\frac{1}{2}k^2\tau_3-k\tau_2]}.$$

Comparing formulae (1) and (3) we find that the degree of N_c on F equals $-2np$. Since we have n^2p^2 sections L_{ij} it follows that $b=-2/np$. To determine the coefficient a we have to compute the degree of N_c on the zero section L_{00}. Since $L_{00}^2=-\deg L_{X(np)}$ we must show that this degree is 0. There are two ways to see this. The first is to use formula (2) and specialise it to $\tau_2=0$. One then has to show that this description extends over the cusps which is an easy local calculation. Alternatively one can proceed as follows: The section L_{00} is the transversal intersection of D_c with the closure of the image of the diagonal $\mathbb{H}_1\times\mathbb{H}_1\subset\mathbb{H}_2$ which parametrizes products. This closure is isomorphic to $X(n)\times X(np)$ and L_{00} is equal to $\{\text{cusp}\}\times X(np)$. Hence the normal bundle of L_{00} in $X(n)\times X(np)$ is trivial and by adjunction

$$K_{L_{00}}=K|_{L_{00}}+L_{00}|L_{00}$$

where K is the canonical bundle of $(\mathcal{A}_{1,p}^{\mathrm{lev}}(n))^*$. Using the fact that $K=3L-D$ and pulling this back to $S(np)$ we obtain

$$K_{L_{00}}=\left(3L_{X(np)}-t(np)F-N_c+L_{00}\right)|_{L_{00}}.$$

Since $\deg K_{L_{00}} = t(np)(np-6)/6$ a straightforward calculation shows that the degree of $N_c|_{L_{00}}$ is equal to 0.

The calculation for N_p is essentially the same. The only differences are that $t_3 = e^{2\pi i \tau_3/np^2}$ and that the fibre of $S(n)$ over $[\tau_1] \in X(n)$ is equal to $E_{[\tau,3]} = \mathbb{C}/(\mathbb{Z}np\tau_1 + np\mathbb{Z})$. □

Proof of the theorem: We first remark that $\left(\mathcal{A}^{\mathrm{lev}}_{1,p}(n)\right)^*$ is smooth under the assumptions made. Since $n \geq 4$ is the group $\Gamma^{\mathrm{lev}}_{1,p}(n)$ is neat. Therefore it is enough to show that for a given cusp h the toroidal variety $T_{\Sigma_{\mathrm{vor}}}(P'_{\Gamma^{\mathrm{lev}}_{1,p}(n)}(h))$ is smooth. If n and p are coprime, the lattice $P'_{\Gamma^{\mathrm{lev}}_{1,p}(n)}(h)$ is simply n times the corresponding lattice in $\Gamma^{\mathrm{lev}}_{1,p}$. Hence every $\sigma \in \Sigma_{\mathrm{vor}}$ is spanned over $\mathbb{R}$ by a basis of the lattice and this implies that T_σ is smooth.
The next observation is that the condition $n \geq 5$ is necessary. We have already remarked that the closure of the diagonal $\mathbb{H}_1 \times \mathbb{H}_1 \subset \mathbb{H}_2$ parametrizing split abelian surfaces is isomorphic to $X(n) \times X(np)$. Consider a curve $C = X(n) \times \{\text{point}\}$. Then

$$K|_C = (3L - D)|_C = 3L_{X(n)} - X_\infty(n)$$

where $X_\infty(n)$ is the divisor of cusps on $X(n)$. Hence

$$K.C = \frac{n}{4}t(n) - t(n)$$

and this is positive if and only if $n \geq 5$.
We shall now assume $n \geq 5$. Let C be an irreducible curve which is not entirely contained in the boundary, i.e. $C \cap \mathcal{A}^{\mathrm{lev}}_{1,p}(n) \neq \emptyset$ and consider a point $[\tau] \in C$. Choose some $\varepsilon > 0$ with $\varepsilon < 3n/5$. By Weissauer's result [We, p. 220] we can find a cusp form F with respect to $\mathrm{Sp}(4,\mathbb{Z})$ such that $F(\tau) \neq 0$ and $o(F) \geq 1/(12+\varepsilon)$. Here $o(F)$ is the order of F, i.e. the vanishing order of F divided by the weight of F. Let m and k be the vanishing order, resp. the weight of F. Since $\Gamma^{\mathrm{lev}}_{1,p}(n) \subset \mathrm{Sp}(4,\mathbb{Z})$ the form F is also a modular form with respect to $\Gamma^{\mathrm{lev}}_{1,p}(n)$. In terms of divisors this gives us

$$kL = mnD + D_{\mathrm{eff}}, \quad C \not\subset D_{\mathrm{eff}}.$$

Here D_{eff} contains in particular multiples of peripheral boundary components since the vanishing order of F along these boundary components is at least np^2. From the above formula we find that

$$\left(\frac{k}{mn}L - D\right).C = \frac{1}{mn}D_{\mathrm{eff}}.C \geq 0.$$

Since $L.C > 0$ we find that $K.C > 0$ provided $3 > k/mn$. But this follows immediately from the inequalities $m/k \geq 1/(12+\varepsilon)$ and $\varepsilon < 3n/5$.
It remains to prove that the restriction of K to every boundary component

is ample. Let D_0 be a boundary component and set $D_0' = D - D_0$. We have already observed that there is an immersion $f : \tilde{D} \to D_0$ which is the normalization. The surface $\tilde{D}$ is either isomorphic to $S(np)$ or to $S(n)$ depending on whether we have a central or a peripheral boundary component. The map f embeds every component of a singular fibre. The image of such a component in $\left(\mathcal{A}_{1,p}^{\mathrm{lev}}(n)\right)^*$ is a $\mathbb{P}^1$. Away from $\{0, \infty\}$ this line is either the intersection of 2 different boundary components or 2 branches of D_0 intersecting transversally. In either case we have the

$$f^*K = f^*(3L - D_0' - D_0) = 3L_X - F_\infty - N_f.$$

Here L_X is either $L_{X(np)}$ or $L_{X(n)}$ depending on the type of the boundary component, the divisor F_∞ is the union of the singular fibres and N_f is the normal bundle of the immersion f. Let $k = np$ or n. Then

$$\deg(3L_X - F_\infty) = \frac{1}{4}kt(k) - t(k) > 0$$

for $n > 4$. Hence $(3L_X - F_\infty).C \geq 0$ for every curve C and $(3L_X - F_\infty).C > 0$ unless C is contained in a union of fibres. It follows immediately from our proposition that $-N_f.C > 0$ for every curve C which does not contain a section L_{ij}. Since $-N_f.L_{ij} = 0$ we can conclude that $f^*K.C > 0$ for every curve C. $\square$

Remark The above proof can also be adapted to show that K is nef for $n = 4$. We had already seen that K is not ample in this case. In other words $\left(\mathcal{A}_{1,p}^{\mathrm{lev}}(4)\right)^*$ is a minimal, but not a canonical model for $p = 1$ or $p \geq 3$ prime.

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